Digital Communication Systems

ECS 452

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7. The Waveform Channel

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The Big Plan

- Although we are thinking about digital communication systems,
- actual signaling in the wire or air is in continuous time which is described by the **waveform channel**:
$$R(t) = S(t) + N(t).$$
- Directly finding the optimal (MAP) detector or evaluating the performance $P(\mathcal{E})$ of such system is difficult.
- Our approach is to first construct an equivalent **vector channel** that preserves the relevant features. (Chapter 7)
- Then, at the end, we can use what we learn to go back to the original waveform channel and directly work with the waveforms.

Review: ECS315

ECS 315: Probability and Random Processes **2018/1**

HW Solution 11 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 2 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

[ECS315 2018]

Autocorrelation Function

For random variables, correlation: $\mathbb{E}[XY]$

expectation operator

11.4 Linear Dependence

Definition 11.47. Given two random variables X and Y, we may calculate the following quantities:

(a) **Correlation**: $\mathbb{E}[XY]$.

(b) **Covariance**: $\text{Cov}[X,Y] = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$. $= \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y$

(c) **Correlation coefficient**: $\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}$ $\mathbb{E}\left[\left(\frac{X-\mathbb{E}X}{\sigma_X}\right)\left(\frac{Y-\mathbb{E}Y}{\sigma_Y}\right)\right]$

[ECS315 2018]

For a random process, autocorrelation function:

$$R_X(t_1,t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right]$$

For two random processes, cross-correlation function:

$$R_{X,Y}(t_1,t_2) = \mathbb{E}\left[X(t_1)Y(t_2)\right]$$

Review: ECS315

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution:

(b) Y is another function of Θ.

$$\mathbb{E}[Y] = \mathbb{E}[5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1 + \theta) \times 5\cos(7t_2 + \theta) d\theta$$
$$= \frac{25}{2\pi} \int_0^{2\pi} \cos(7t_1 + \theta) \times \cos(7t_2 + \theta) d\theta.$$

Recall[1] the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$$

Therefore,

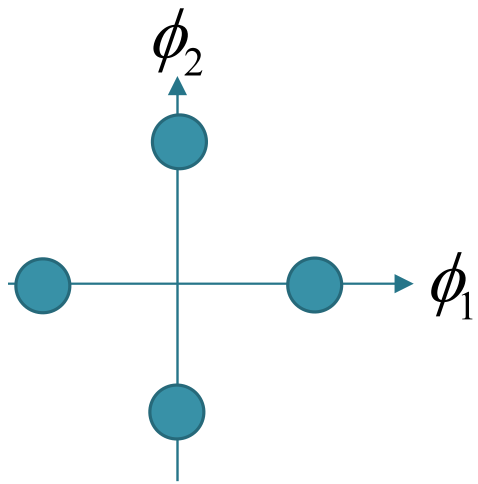
$$\mathbb{E}Y = \frac{25}{4\pi} \int_0^{2\pi} \cos(7t_1 + 7t_2 + 2\theta) + \cos(7(t_1 - t_2))\, d\theta$$
$$= \frac{25}{4\pi} \left(\int_0^{2\pi} \cos(7t_1 + 7t_2 + 2\theta)\, d\theta + \int_0^{2\pi} \cos(7(t_1 - t_2))\, d\theta \right).$$

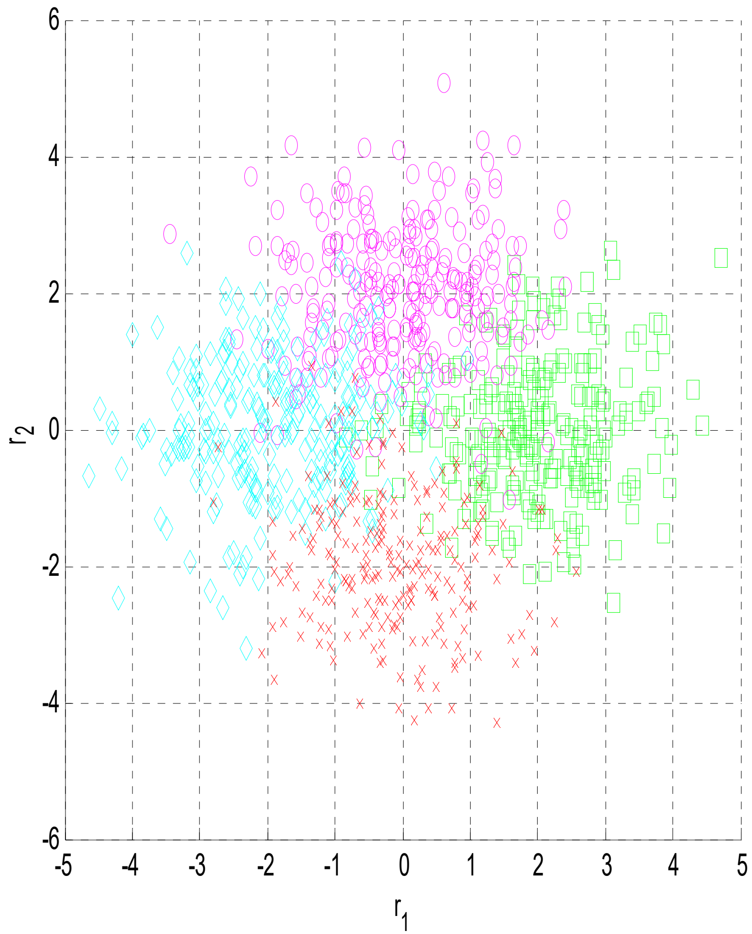
The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi} \cos(7(t_1 - t_2)) \int_0^{2\pi} d\theta = \frac{25}{4\pi} \cos(7(t_1 - t_2))\, 2\pi = \boxed{\frac{25}{2} \cos(7(t_1 - t_2))}.$$

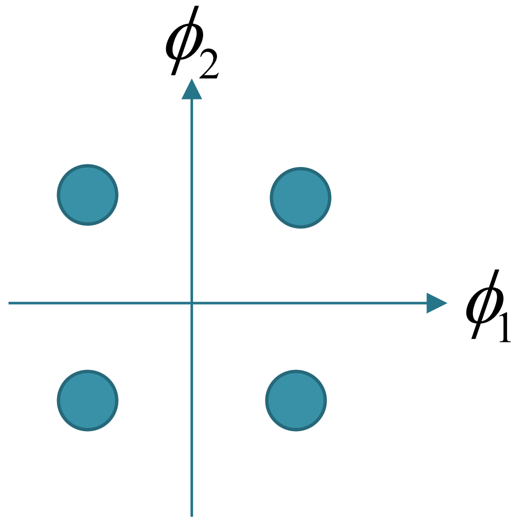
[ECS315 2018]

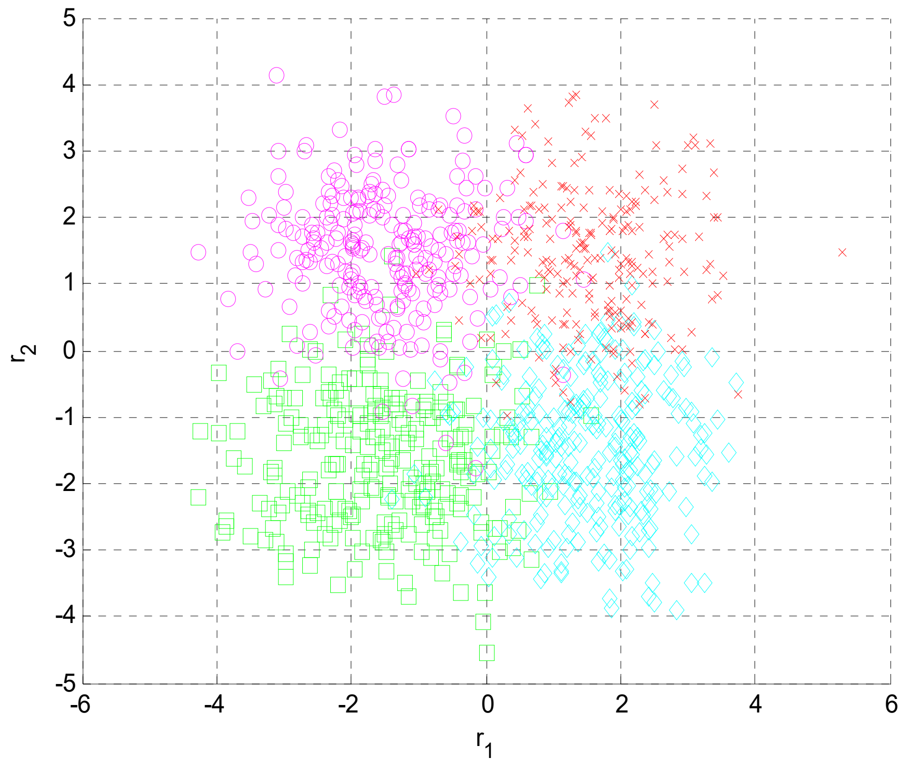
Standard Quaternary PSK





Standard Quaternary QAM





Modulator and Waveform Channel

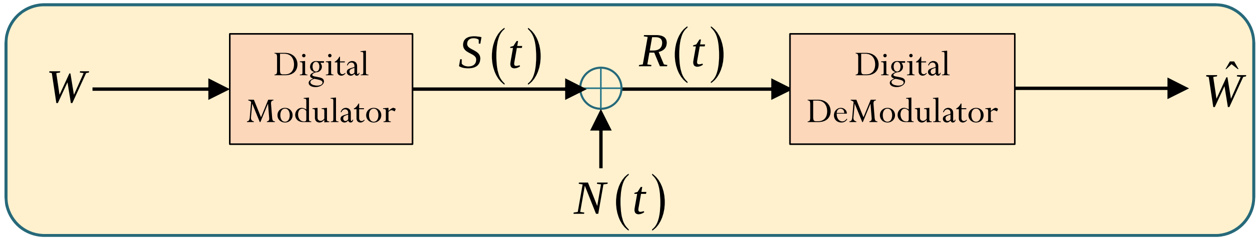
Goal: Want to transmit the message (index) $W \in \{1, 2, 3, \ldots, M\}$

Prior Probabilities: $p_j = P[W = j]$

***M*-ary** Scheme

M =2: Binary
M = 3: Ternary
M = 4: Quaternary

Waveform Channel:



M possible messages requires *M* possibilities for *S(t)*:

$$\{s_1(t), s_2(t), \ldots, s_M(t)\}$$

$$R(t) = S(t) + N(t)$$

Additive White Noise (Independent of $S(t)$)

Transmitted waveform

Received waveform

Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.

Prior Probabilities: $p_j = P[W = j] = P[S(t) = s_j(t)]$

Energy: $E_j = \langle s_j(t), s_j(t) \rangle \qquad E_s = \sum_{j=1}^{M} p_j E_j = (\log_2 M) E_b$

Conversion to Vector Channels

Waveform Channel: $R(t) = S(t) + N(t)$

Find K orthonormal basis functions $\{\phi_1(t), \phi_2(t), \ldots, \phi_K(t)\}$ for the space spanned by $\{s_1(t), s_2(t), \ldots, s_M(t)\}$. This gives vector representations for the waveforms $s_1(t), s_2(t), \ldots, s_M(t)$:

$$\vec{\mathbf{s}}^{(1)}, \vec{\mathbf{s}}^{(2)}, \ldots, \vec{\mathbf{s}}^{(M)}$$

which can be visualized in the form of signal constellation

Vector Channel $\vec{\mathbf{R}} = \vec{\mathbf{S}} + \vec{\mathbf{N}}$

Note that $S_i^{(j)}$, the i^{th} component of the $\vec{\mathbf{S}}$ vector, comes from the inner-product:

$$S_i^{(j)} = \langle S(t), \phi_i(t) \rangle$$

The received vector $\vec{R}$ is computed in the same way: the j component is given by

$$R_i = \langle r(t), \phi_i(t) \rangle$$

In which case, the corresponding noise vector $\vec{\mathbf{N}}$ is computed in the same way: the j component is given by

$$N_i = \langle N(t), \phi_i(t) \rangle$$

Prior Probabilities:

$$p_j = P[W = j] = P[S(t) = s_j(t)]$$

$$= P\left[\vec{\mathbf{S}} = \vec{\mathbf{s}}^{(j)}\right]$$

For additive white **Gaussian** noise (AWGN) process $N(t)$,

$$N_i \sim N \sim \mathcal{N}\left(0, \frac{N_0}{2}\right) = \mathcal{N}(0, \sigma^2) \Rightarrow \vec{\mathbf{N}} \sim \mathcal{N}\left(\vec{\mathbf{0}}, \frac{N_0}{2} I\right) \Rightarrow f_{\vec{\mathbf{N}}}(\vec{\mathbf{n}}) = \frac{1}{(2\pi)^{\frac{K}{2}} \sigma^K} e^{-\frac{1}{2}\frac{\|\vec{\mathbf{n}}\|^2}{\sigma^2}}$$